On the period of the continued fraction for values of the square root of power sums

Amedeo SCREMIN*

February 1, 2008

Abstract

The present paper proves that if for a power sum α over $\mathbb{Z}$ the length of the period of the continued fraction for $\sqrt{\alpha(n)}$ is constant for infinitely many even (resp. odd) n, then $\sqrt{\alpha(n)}$ admits a functional continued fraction expansion for all even (resp. odd) n, except finitely many; in particular, for such n, the partial quotients can be expressed by power sums of the same kind.

1 Introduction

It is well known that the continued fraction for rational numbers is finite and that for the square root of a positive integer a which is not a square is periodic of the form $[a_o; \overline{a_1, \ldots, a_{R-1}, 2a_0}]$ (here with $\overline{a_1, \ldots, a_{R-1}, 2a_0}$ we denote the periodic part), where $R \geq 1$ is the length of the period. About R, we know that the bound $R \ll \sqrt{a} \log a$ holds (see [4] and [6]).

A power sum α is a function on $\mathbb{N}$ of the form

$$\alpha(n) = b_1 c_1^n + b_2 c_2^n + \ldots + b_h c_h^n, \tag{1}$$

where the roots c_i are integers and the coefficients b_i are in $\mathbb{Q}$ or in $\mathbb{Z}$. We know from Corollary 1 in [2] that, apart from the case when α is the square of a power sum of the same kind, $\sqrt{\alpha(n)}$ is a quadratic irrational for all $n \in \mathbb{N}$, except finitely many. This means that the continued fraction expansion for

*The author was supported by Istituto Nazionale di Alta Matematica "Francesco Severi", grant for abroad Ph.D

$\sqrt{\alpha(n)}$ is periodic for n large, raising the problem weather the length of the period is bounded or not for $n \longrightarrow +\infty$, which will be considered in this paper. Some partial results on such problem have been recently obtained by Bugeaud and Luca (see [1]).

On a similar problem, but considering a non constant polynomial f with rational coefficients instead of the power sum α, remarkable results were obtained by Schinzel in [7] and [8]. He provided conditions on f under which the length of the period of the continued fraction for $\sqrt{f(n)}$ tends to infinity as $n \longrightarrow +\infty$.

In the present paper we shall first prove that if a power sum α with rational coefficients cannot be approximated "too well" by the square of a power sum of the same kind, then the length of the period of the continued fraction for $\sqrt{\alpha(n)}$ tends to infinity as $n \longrightarrow +\infty$ (Corollary 3.3).

Then we shall consider power sums with integral coefficients, and show that for any fixed $r \in \{0,1\}$, if the length of the period of the continued fraction for $\sqrt{\alpha(2m+r)}$ is constant for all m in an infinite set, then for every $m \in \mathbb{N}$, except finitely many exceptions, the partial quotients of the continued fraction for $\sqrt{\alpha(2m+r)}$ can be identically expressed by power sums of the same kind (Main Theorem 3.4).

The results above shall be deduced from some lower bounds for the quantities $|\sqrt{\alpha(n)} - \frac{p}{q}|$ (Corollary 3.2) and $\left|\frac{\sqrt{\alpha(n)}+\beta(n)}{\gamma(n)} - \frac{p}{q}\right|$ (Theorem 3.1) respectively, where α, β, γ are power sums and p, q are integers, which we shall obtain using Schmidt's Subspace Theorem in a way similar to that of Corvaja and Zannier in [2] and [3].

Theorem 3.1 and Corollary 3.2 (taking $\alpha = 0$ and $q = 1$ respectively), are the analogue of the Theorem in [3] and of Theorem 3 in [2].

The results contained in this paper give an answer to some questions raised in the Final Remark (b) in [3], where it is predicted that "under suitable assumptions on the power sum α with rational roots and coefficients, the length of the period of the continued fraction for $\sqrt{\alpha(n)}$ tends to infinity with n".

2 Notation

In the present paper we will denote by Σ the ring of functions on $\mathbb{N}$, called *power sums*, of the form

$$\alpha(n) = b_1 c_1^n + b_2 c_2^n + \ldots + b_h c_h^n, \tag{2}$$

where the distinct *roots* $c_i \neq 0$ are in $\mathbb{Z}$, and the *coefficients* $b_i \in \mathbb{Q}^\star$. For rings $A, B \subseteq \mathbb{C}$, let $A\Sigma_B$ denote the ring of power sums with coefficients in A and roots in B.

If $B \subseteq \mathbb{R}$, it is usually enough to deal with power sums with only positive roots. Working in this domain causes no loss of generality: the assumption of positivity of the roots may usually be achieved by writing $2n + r$ instead of n, and considering the cases of $r = 0, 1$ separately.

If $\alpha \in \overline{\mathbb{Q}}\Sigma_{\mathbb{Q}}$, we set $l(\alpha) := \max\{c_1, \ldots, c_h\}$. In the same way we define the function l for a power sum defined on the sets of even or odd numbers. It is immediate to check that $l(\alpha\beta) = l(\alpha)l(\beta)$, $l(\alpha + \beta) \leq \max\{l(\alpha), l(\beta)\}$ and that $l(\alpha)^n \gg |\alpha(n)| \gg l(\alpha)^n$.

NOTE In the statements of our results and in the proofs we will always omit the condition for the existence of $\sqrt{\alpha(n)} \in \mathbb{R}$, i.e. that $\alpha(n) \geq 0$ for n large.

3 Statements

The following Theorem 3.1 states that for power sums $\alpha, \beta, \gamma \in \Sigma$, if $\dfrac{\sqrt{\alpha} + \beta}{\gamma}$ cannot be well approximated on the subsequence of even (or odd) numbers by a power sum in Σ, then $\dfrac{\sqrt{\alpha(n)} + \beta(n)}{\gamma(n)}$ cannot be well approximated by rationals with exponentially bounded denominators, except for a finite number of even (odd) n. This Diophantine approximation result will be obtained using Schmidt's Subspace Theorem in a way similar to that of Corvaja and Zannier in [2] and [3]. Theorem 3.1 is the main tool we will use to prove the Corollaries and the Main Theorem.

Theorem 3.1 *Let* $\alpha, \beta, \gamma \in \Sigma$, γ *not identically zero, and let* $\varepsilon > 0$ *and* $r \in \{0, 1\}$ *be fixed.*

Suppose that there does not exist a power sum $\eta \in \Sigma$ *such that*

$$\Big| \frac{\sqrt{\alpha(2m+r)} + \beta(2m+r)}{\gamma(2m+r)} - \eta(m) \Big| \ll e^{(-2m+r)\epsilon}.$$

Then there exist $k = k(\alpha, \beta, \gamma) > 2$ *and* $Q = Q(\epsilon) > 1$ *with the following properties. For all but finitely many naturals* $n \equiv r \mod 2$ *and for integers* $p, q,\ 0 < q < Q^{2m+r}$, *we have*

$$\Big| \frac{\sqrt{\alpha(n)} + \beta(n)}{\gamma(n)} - \frac{p}{q} \Big| \geq \frac{1}{q^k} e^{-\epsilon n}. \tag{3}$$

Remark 1 Taking $\alpha = 0$ in Theorem 3.1, we obtain again the result of the Theorem in [3].

Corollary 3.2 is a simplified version of Theorem 3.1. It states that if a power sum $\alpha \in \Sigma$ cannot be well approximated on the subsequences of even and odd numbers by the square of a power sum from the same ring, then $\sqrt{\alpha(n)}$ cannot be well approximated by rationals with exponentially bounded denominators, except for a finite number of n. It will be used to prove Corollary 3.3.

Corollary 3.2 *Let* $\alpha \in \Sigma$, *and let* $\varepsilon > 0$ *be fixed. Assume that for every* $r \in \{0, 1\}$ *and for all* $\xi \in \Sigma$,

$$l(\alpha - \xi^2) \geq l(\alpha)^{1/2}$$

on the sequence $n = 2m + r$.

Then there exist $k = k(\alpha) > 2$ *and* $Q = Q(\epsilon) > 1$ *with the following properties. For all but finitely many* $n \in \mathbb{N}$ *and for all integers* $p, q,\ 0 < q < Q^n$, *we have*

$$\Big| \sqrt{\alpha(n)} - \frac{p}{q} \Big| \geq \frac{1}{q^k} e^{-\epsilon n}. \tag{4}$$

Remark 2 Taking $q = 1$, we can see that Corollary 3.2 is a generalization of Theorem 3 in [2].

Remark 3 In concrete cases, it is easy to verify whether the assumption of Corollary 3.2 holds or not. In fact, it is enough to prove that for every $r \in \{0,1\}$ and for all $\eta \in \Sigma$, in the power sum $\alpha(2m+r) - \eta(m)^2$ there cannot be cancellations of all the coefficients of the roots greater than the square root of the dominating root of α (resp., there exists η such that we have all that cancellations). By a similar way it is possible to verify if the assumption of Theorem 3.1 holds or not.

The following Corollary 3.3 states that if a power sum $\alpha \in \Sigma$ cannot be well approximated by the square of a power sum of the same kind, then the length of the period of the continued fraction for $\sqrt{\alpha(n)}$ tends to infinity as $n \longrightarrow +\infty$. This result was already obtained with a similar proof by Bugeaud and Luca in [1, Theorem 2.1].

Corollary 3.3 *Let $\alpha \in \Sigma$ be as in the Corollary 3.2. Then the length of the period of the continued fraction for $\sqrt{\alpha(n)}$ tends to infinity as $n \to +\infty$.*

The Main Theorem 3.4 follows again from Theorem 3.1, and states that if the length of the period of the continued fraction for the square root of a power sum is constant for infinitely many even (resp. odd) n, then the partial quotients of the continued fraction can be expressed by power sums for all even (resp. odd) n, except finitely many.

Main Theorem 3.4 *Let $\alpha \in \mathbb{Z}\Sigma_{\mathbb{Z}}$, and let $r \in \{0,1\}$ be fixed. Suppose that there exists an infinite set $A \subseteq \mathbb{N}$ and a constant $R \geq 0$ such that for $m \in A$ the length of the period of the continued fraction expansion for $\sqrt{\alpha(2m+r)}$ is R. Then there exist $\beta_0, \ldots, \beta_R \in \mathbb{Z}\Sigma_{\mathbb{Z}}$ such that for every $m \in \mathbb{N}$, apart from finitely many exceptions, we have the continued fraction expansion*

$$\sqrt{\alpha(2m+r)} = [\beta_0(m); \overline{\beta_1(m), \ldots, \beta_R(m)}]. \tag{5}$$

Remark 4 The result of Corollary 3.3, together with the Main Theorem 3.4, gives an answer to the question raised in the Final Remark (b) in [3].

4 Auxiliary results

For the reader's convenience we state here a version of Schmidt's Subspace Theorem due to H.P. Schlickewei; we have borrowed it from [10, Theorem 1E, p. 178] (a complete proof requires also [9]). It will be our main tool to prove Theorem 3.1.

Theorem 4.1 *Let S be a finite set of absolute values of $\mathbb{Q}$, including the infinite one and normalized in the usual way (i.e. $|p|_v = p^{-1}$ if $v|p$). Extend each $v \in S$ to $\overline{\mathbb{Q}}$ in some way. For $v \in S$ let $L_{1,v}, \ldots, L_{n,v}$ be n linearly independent linear forms in n variables with algebraic coefficients and let $\delta > 0$.*
Then the solutions $\underline{x} := (x_1, \ldots, x_n) \in \mathbb{Z}^n$ to the inequality

$$\prod_{v \in S} \prod_{i=1}^{n} |L_{i,v}(\underline{x})|_v < \max_{1 \leq i \leq n} |x_i|^{-\delta}$$

are contained in finitely many proper subspaces of $\mathbb{Q}^n$.

The following Lemma 4.2 is a result by Evertse (in a more general case); a proof by Corvaja and Zannier can be found in [2, Lemma 2].

Lemma 4.2 *Let $\xi \in \Sigma_{\mathbb{Q}}$ and let D be the minimal positive integer such that $D^n \xi \in \Sigma$.*
Then, for every $\varepsilon > 0$, there are only finitely many $n \in \mathbb{N}$ such that the denominator of $\xi(n)$ is smaller than $D^n e^{-n\varepsilon}$.

5 Proofs

We start with the following very simple

Lemma 5.1 *Let* $\alpha, \beta, \gamma \in \Sigma$, γ *not identically zero, and let* t *be any positive real number. Then for every* $r \in \{0,1\}$ *there exists* $\eta_r \in \overline{\mathbb{Q}}\Sigma_{\mathbb{Q}}$ *such that*

$$\Big|\frac{\sqrt{\alpha(2m+r)}+\beta(2m+r)}{\gamma(2m+r)} - \eta_r(2m+r)\Big| \ll t^{2m}.$$

Such η_r *can be effectively computed in terms of* r, α, β, γ *and* t.

Proof of Lemma 5.1. Let $\alpha(n) = \sum_{j=1}^{h} b_j\, c_j^n$, with $c_j \in \mathbb{Z}$, $c_j \neq 0$ and $b_j \in \mathbb{Q}^*$ $\forall\, j = 1, \ldots, h$.

We can suppose $c_1 > c_2 > \ldots > c_h > 0$.

For a real determination (resp. real positive) of $b_1^{1/2}$ (resp. $c_1^{1/2}$), fixed for the rest of the proof, we have

$$\alpha(n)^{1/2} = (b_1 c_1^n)^{1/2}\Big(1 + \sum_{j=2}^{h} \frac{b_j}{b_1}\big(\frac{c_j}{c_1}\big)^n\Big)^{1/2} \;=\; (b_1 c_1^n)^{1/2}(1+\sigma(n))^{1/2}, \qquad (6)$$

with $\sigma(n) \in \Sigma_{\mathbb{Q}}$, and $\sigma(n) \,=\, O((c_2/c_1)^n)$.

Expanding the function $x \mapsto (1+x)^{1/2}$ in Taylor series, we have

$$(1+\sigma(n))^{1/2} \;=\; 1 + \sum_{j=1}^{H} a_j\; \sigma(n)^j + O(|\sigma(n)|^{H+1}), \qquad (7)$$

where $H > 0$ is an integer that can be chosen later and a_j, $j = 1, \ldots, H$, are the Taylor coefficients $\binom{1/2}{j}$ of the function $x \mapsto (1+x)^{1/2}$.

For every $r \in \{0,1\}$, substituting (7) in (6) we obtain

$$\alpha(2m+r)^{1/2} = b_1^{1/2} c_1^{r/2} c_1^m \Big(1 + \sum_{j=1}^{H} a_j \sigma(2m+r)^j\Big) + O\Big(\Big(\frac{c_2}{c_1}\Big)^{2m(H+1)} c_1^m\Big). \; (8)$$

Let

$$\beta(n) = \sum_{j=1}^{k} d_j e_j^n \in \Sigma, \qquad (9)$$

with $e_j \in \mathbb{Z}$, $e_j \neq 0$ and $d_j \in \mathbb{Q}^*$ $\forall\, j = 1, \ldots, h$.

We can suppose $e_1 > e_2 > \ldots > e_k > 0$.

Fix H such that $\Big(\frac{c_2}{c_1}\Big)^{(H+1)} c_1^{1/2} < e_1$.

Let $\gamma(n) = \sum_{j=1}^{l} f_j g_j^n \in \Sigma$, with $g_j \in \mathbb{Z}$, $g_j \neq 0$ and $f_j \in \mathbb{Q}^*$ $\forall\, j = 1, \ldots, h$.

We can suppose $g_1 > g_2 > \ldots > g_k > 0$.

Using the same method as in the proof of Theorem 1 in [2], we can write

$$\gamma(n)^{-1} = f_1^{-1} g_1^{-n} \sum_{j=0}^{s} \phi(n)^j + O((g_2/g_1)^{n(s+1)} g_1^{-n}), \tag{10}$$

where $\phi(n) := -\sum_{i=2}^{l} \frac{f_i}{f_1} \left(\frac{g_i}{g_1}\right)^n \in \Sigma_{\mathbb{Q}}$, $\phi(n) = O(g_2/g_1)^n$, and $s > 0$ is an integer that can be chosen later.

Thus, by equations (8), (9), (10), by the choice of H and the definition of ϕ, we obtain

$$\frac{\sqrt{\alpha(2m+r)} + \beta(2m+r)}{\gamma(2m+r)} = f_1^{-1} g_1^{-r} g_1^{-2m} \Big(\sum_{i=0}^{s} \phi(2m+r)^i \Big) \cdot$$
$$\cdot \Big(b_1^{1/2} c_1^{r/2} c_1^m \big(1 + \sum_{i=1}^{H} a_i \sigma(2m+r)^i\big) + \sum_{i=1}^{k} d_i e_i^{2m+r} \Big) + O\Big(\big(g_2/g_1\big)^{2m(s+1)} g_1^{-2m} e_1^{2m} \Big).$$

Fix s such that $\big(g_2/g_1\big)^{(s+1)} g_1^{-1} e_1 < t$ and put, for $r = 0, 1$,

$$\eta_r(2m+r) := f_1^{-1} g_1^{-r} g_1^{-2m} \Big(\sum_{i=0}^{s} \phi(2m+r)^i \Big) \cdot$$
$$\cdot \Big(b_1^{1/2} c_1^{r/2} c_1^m \big(1 + \sum_{i=1}^{H} b_i \sigma(2m+r)^i\big) + \sum_{i=1}^{k} d_i e_i^{2m+r} \Big).$$

By definition $\eta_r \in \overline{\mathbb{Q}}\Sigma_{\mathbb{Q}}$ for every $i = 0, 1$.

Thus for every $r \in \{0, 1\}$ we have effectively constructed a power sum $\eta_r(n) \in \overline{\mathbb{Q}}\Sigma_{\mathbb{Q}}$ such that

$$\Big| \frac{\sqrt{\alpha(2m+r)} + \beta(2m+r)}{\gamma(2m+r)} - \eta_r(2m+r) \Big| \ll t^{2m},$$

completing the proof.

$\square$

Remark 5 Let us notice that in η_r the root with largest absolute value is $g_1^{-2} \cdot \max\{e_1^2, c_1\}$ and that the other roots appearing are rational with denominator powers of c_1 and g_1. The denominators of each of such roots are divided by g_1^2.

Proof of Theorem 3.1. Let η_r, for $r \in \{0, 1\}$ fixed, be as in Lemma 5.1, with $t = 1/9$.

We can write (recall Remark 6)

$$\eta_r(2m+r) = b_{1,r}^{1/2} d_1^m \big(g^{-2m} + b_2 d_2^{2m+r} + \ldots + b_h d_h^{2m+r}\big),$$

for some $b_{1,r}, b_i \in \overline{\mathbb{Q}}^*$, $d_1, g \in \mathbb{Z}\setminus\{0\}$, $d_2,\ldots,d_h \in \mathbb{Q}$, $g^{-2} > d_2 > \ldots > d_h > 0$. We define $k := h+3$ and, for the $\epsilon > 0$ fixed (which we may take $< 1/2k$, say), $Q = e^{\epsilon}$. We suppose that there are infinitely many triples (m,p,q) of integers with $0 < q < Q^{2m+r}$, $m \to +\infty$ and

$$\Big|\frac{\sqrt{\alpha(2m+r)} + \beta(2m+r)}{\gamma(2m+r)} - \frac{p}{q}\Big| \leq \frac{1}{q^k} e^{-\epsilon(2m+r)}. \tag{11}$$

We shall eventually obtain a contradiction, which will prove what we want.

We proceed to define the data for an application of the Subspace Theorem 4.1. We let S be the finite set of places of $\mathbb{Q}$ containing the infinite one and all the places dividing the numerators or the denominators of g and of d_i, $i = 1,\ldots,h$. We define linear forms in $X_o,\ldots,X_h$ as follows. For $v \neq \infty$ or for $i \neq 0$ we set simply $L_{i,v} = X_i$. We define the remaining form

$$L_{0,\infty} := X_0 - b_{1,r}^{1/2} X_1 - b_{2,r} X_2 - \ldots - b_{h,r} X_h,$$

where $b_{i,r} = b_i b_{1,r}^{1/2}$, $i = 2,\ldots,h$. For each v, these linear forms are clearly independent.

Let d be the minimal integer such that $d_i d \in \mathbb{Z}$ for every $i = 1,\ldots,h$ (recall Remark 6). For our choice of the set S, d is a S-unit.

Define $e_1 := d_1 d g^{-2}$, $e_i := d d_i$, $i = 2,\ldots,h$. Note that $e_i \in \mathbb{Z}$ for every $i = 1,\ldots,h$.

Set the vector

$$\underline{x} = \underline{x}(m,p,q) = (p d^{2m+r},\ q e_1^m d^{m+r},\ q d_1^m e_2^{2m+r},\ \ldots,\ q d_1^m e_h^{2m+r}) \in \mathbb{Z}^{h+1}.$$

We proceed to estimate the double product $\prod_{v\in S}\prod_{i=0}^{h} |L_{i,v}(\underline{x})|_v$.

We have

$$\prod_{v\in S}\prod_{i=0}^{h} |L_{i,v}(\underline{x})|_v = |L_{0,\infty}(\underline{x})| \cdot \prod_{i=1}^{h}\prod_{v\in S} |L_{i,v}(\underline{x})|_v \cdot \prod_{v\in S\setminus\{\infty\}} |L_{0,v}(\underline{x})|_v. \tag{12}$$

By definition $\prod_{v\in S}|L_{1,v}(\underline{x})|_v = \prod_{v\in S}|qe_1^m d^{m+r}|_v \leq q$ and, for $i \geq 2$, $\prod_{v\in S}|L_{i,v}(\underline{x})|v = \prod_{v\in S}|qd_1^m e_i^{2m+r}|_v \leq q$, since d, d_1 and the e_i are S-units for every i (which implies that $\prod_{v\in S}|d|_v = \prod_{v\in S}|d_1|_v = \prod_{v\in S}|e_i|_v = 1$) and since for the positive integer q, $\prod_{v\in S}|q|_v \leq q$ holds. This means that

$$\prod_{i=1}^{h}\prod_{v\in S}|L_{i,v}(\underline{x})|_v \leq q^h. \tag{13}$$

Moreover,

$$\prod_{v\in S\setminus\{\infty\}}|L_{0,v}(\underline{x})|_v = \prod_{v\in S\setminus\{\infty\}}|pd^{(2m+r)}|_v =$$

$$= \prod_{v\in S\setminus\{\infty\}}|p|_v \cdot \prod_{v\in S\setminus\{\infty\}}|d^{(2m+r)}|_v \leq d^{-(2m+r)}, \tag{14}$$

the last inequality holding since p is an integer and d is a S-unit.

Finally we have

$$|L_{0,\infty}(\underline{x})| = d^{2m+r}\big|p - q\big(b_{1,r}^{1/2} d_1^m g^{-2m} + b_{2,r} d_1^m d_2^{2m+r} + \ldots + b_{h,r} d_1^m d_h^{2m+r}\big)\big| =$$
$$= qd^{2m+r}\Big|\eta_r(2m+r) - \frac{p}{q}\Big|,$$

which, combined with (12), (13) and (14), gives

$$\prod_{v\in S}\prod_{i=0}^{h}|L_{0,v}(\underline{x})|_v \leq q^{h+1}\Big|\eta_r(2m+r) - \frac{p}{q}\Big|. \tag{15}$$

Since $q^k < Q^{k(2m+r)} = e^{(2m+r)k\epsilon}$, we have $q^{-k}e^{-(2m+r)\epsilon} > e^{-(2m+r)(k+1)\epsilon}$, which means that $q^{-k}e^{-(2m+r)\epsilon} > t^{2m+r}$ (recall that $\epsilon < 1/2k$, $k \geq 3$ and $t = 1/9$). Thus, for a certain constant $l > 0$, we have

$$\Big|\eta_r(2m+r) - \frac{p}{q}\Big| \leq \Big(\Big|\frac{p}{q} - \frac{\sqrt{\alpha(2m+r)} + \beta(2m+r)}{\gamma(2m+r)}\Big| +$$
$$+ \Big|\frac{\sqrt{\alpha(2m+r)} + \beta(2m+r)}{\gamma(2m+r)} - \eta_r(2m+r)\Big|\Big) \leq \Big(\frac{1}{q^k}e^{-(2m+r)\epsilon} + lt^{2m+r}\Big) \leq$$
$$\leq \frac{2}{q^k}e^{-(2m+r)\epsilon}.$$

This means that $\prod_{v\in S}\prod_{i=0}^{h}|L_{0,v}(\underline{x})|_v \leq 2q^{h+1-k}e^{-(2m+r)\epsilon} \leq e^{-(2m+r)\epsilon}$, since we have $k = h+3$. Also, $\max_{0\leq i\leq h}|x_i| \simeq qe_1^m d^{m+r} \leq Q^{2m+r}e_1^m d^{m+r}$.

Hence, choosing $\delta > 0$, $\delta < \dfrac{\epsilon}{\log(Q^2 e_1 d)}$, we get, for m large,

$$\prod_{v\in S}\prod_{i=0}^{h}|L_{0,v}(\underline{x})|_v \le e^{-(2m+r)\epsilon} < (Q^{2m+r}e_1^m d^{m+r})^{-\delta} \le (\max_{0\le i\le h}|x_i|)^{-\delta},$$

i.e. the inequality of the Subspace Theorem 4.1 is verified.

This implies that the vectors

$$\underline{x} = \underline{x}(m,p,q) = (pd^{2m+r},\ qe_1^m d^{m+r},\ qd_1^m e_2^{2m+r},\ \ldots,\ qd_1^m e_h^{2m+r}) \in \mathbb{Z}^{h+1}$$

are contained in a finite set of proper subspaces of $\mathbb{Q}^{h+1}$. In particular, there exists a fixed subspace, say of equation $z_o X_o - z_1 X_1 - \ldots - z_h X_h = 0$, $z_i \in \mathbb{Q}$, containing an infinity of the vectors in question. We cannot have $z_0 = 0$, since this would entail $z_1 e_1^m d^{m+r} + z_2 d_1^m e_2^{2m+r} + \ldots + z_h d_1^m e_h^{2m+r} =$

$$= d_1^m d^{2m+r}(z_1 g^{-2m} + z_2 d_2^{2m+r} + \ldots + z_h d_h^{2m+r}) = 0$$

for an infinity of m; in turn, the fact that g^{-1} and the d_i are pairwise distinct would imply $z_i = 0$ for all i, a contradiction.

Therefore we can suppose that $z_0 = 1$, and we find that, for the m corresponding to the vectors in question,

$$\frac{p}{q} = d_1^m\Big(z_1 g^{-2m} + \sum_{i=2}^{h} z_i d_i^{2m+r}\Big) =: \xi(m) \ \in \mathbb{Q}\Sigma_{\mathbb{Q}}. \tag{16}$$

Let us show that actually $\xi \in \Sigma$. Assume the contrary; then the minimal positive integer D so that $D^m\xi \in \Sigma$ is ≥ 2. But then equation (16) together with Lemma 4.2 implies that $q \gg 2^m e^{-m\epsilon}$. Since this would hold for infinitely many m, we would find $Q \ge q^{\frac{1}{2m}} \ge \sqrt{2}e^{-\epsilon/2}$, a contradiction since $Q = e^\epsilon$, $\epsilon < 1/2k$ and $k \ge 3$.

Therefore $\xi \in \Sigma$.

Substituting (16) in (11) we get that there exists a power sum $\xi \in \Sigma$ such that

$$\Big|\frac{\sqrt{\alpha(2m+r)} + \beta(2m+r)}{\gamma(2m+r)} - \xi(m)\Big| \ll e^{-(2m+r)\epsilon},$$

a contradiction, concluding the proof.

$\square$

Proof of Corollary 3.2. We know that

$$l(\alpha-\xi^2) = l((\sqrt{\alpha}-\xi)(\sqrt{\alpha}+\xi)) \geq l(\alpha)^{1/2}$$

holds for every $\xi \in \Sigma$ by assumption, and that for every $r \in \{0,1\}$

$$|\sqrt{\alpha(2m+r)} + \xi(2m+r)| < 2 \cdot \max\{\sqrt{\alpha(2m+r)},\ |\xi(2m+r)|\}.$$

If for a certain $\xi \in \Sigma$ we have $|\xi(2m+r)| < k \cdot \sqrt{\alpha(2m+r)}$, for some constant $k > 0$, we get that for such $\xi \in \Sigma$,

$$|\sqrt{\alpha(2m+r)} - \xi(2m+r)| > \frac{1}{2}\ \min\left\{1, \frac{1}{k}\right\}.$$

If for a certain $\xi \in \Sigma$ we have $|\xi(2m+r)| \gg \alpha(2m+r)^{\frac{1}{2}(1+\delta)}$, for some $\delta > 0$, we get

$$|\sqrt{\alpha(2m+r)} - \xi(2m+r)| \gg \alpha(2m+r)^{\frac{1}{2}(1+\delta)}.$$

This proves that there does not exist a power sum $\xi \in \Sigma$ and $\epsilon > 0$ such that

$$|\sqrt{\alpha(2m+r)} - \xi(2m+r)| \ll e^{-(2m+r)\epsilon}.$$

Thus we can apply Theorem 3.1 with $\beta = 0$ and $\gamma = 1$, and get the conclusion.

□

Proof of Corollary 3.3. For notation and basic facts about continued fractions we refer to [5] and [9, Ch. I].

Let us suppose by contradiction that there exists an integer $R > 0$ and an infinite set $A \subseteq \mathbb{N}$ such that for $n \in A$ we have $\sqrt{\alpha(n)} = [a_o(n); \overline{a_1(n), \ldots, a_R(n)}]$.

Let $p_i(n)/q_i(n)$, $i = 0, 1, \ldots,$ with $q_0(n) = 1$, be the (infinite) sequence of the convergents of the continued fraction for $\sqrt{\alpha(n)}$. We recall the relation $\left|\sqrt{\alpha(n)} - \frac{p_i(n)}{q_i(n)}\right| < (a_{i+1}(n)q_i(n)^2)^{-1}$, for $i \geq 0$, which implies that

$$a_{i+1}(n) < \left|\sqrt{\alpha(n)} - \frac{p_i(n)}{q_i(n)}\right|^{-1} q_i(n)^{-2} \tag{17}$$

holds for every $i \geq 0$.

Since α satisfies the assumptions for Corollary 3.2, for some $\epsilon > 0$ to be fixed later there exist $k > 2$ and $Q = e^{\epsilon} > 1$ as in the statement.

Define now the increasing sequence $c_0, c_1, \ldots$ by $c_0 = 0$, and $c_{r+1} = (k+1)c_r + 1$, and choose a positive number $\rho < c_R^{-1} \log Q$, so $e^{c_R \rho} < Q$.

Proceeding by induction as in the proof of Corollary 1 in [2], it can be shown that for every $i = 0, \ldots, R$, and for large n, we have $q_i(n) < e^{c_i \rho n}$, which means that $q_i(n) < Q^n$ for every $i = 0, \ldots, R$ and n large. Thus, we can apply Corollary 3.2 with $p = p_i(n)$, $q = q_i(n)$, and $\epsilon > 0$ to be chosen later. Recalling that $Q = e^\epsilon$, from (17) we get that, for all n but finitely many, the inequality

$$a_{i+1}(n) < \left| \sqrt{\alpha(n)} - \frac{p_i(n)}{q_i(n)} \right|^{-1} q_i(n)^{-2} \leq q_i(n)^k e^{n\epsilon} < Q^{kn} e^{n\epsilon} = e^{n(k+1)\epsilon}, \quad (18)$$

holds for every $i = 0, \ldots, R$ and $\epsilon > 0$.

Taking $\delta := (k+1)\epsilon$ we can rewrite the above inequality as

$$a_i(n) < e^{n\delta}, \qquad (19)$$

for $i = 0, \ldots, R$ and for all n but finitely many.

Let us consider from now on $n \in A$ such that $a_i(n) < e^{n\delta}$ holds.

From well known results of the theory of continued fractions (see [5]) we get that for every n,

$$\sqrt{\alpha(n)} = a_0(n) + \frac{1}{\beta(n)}, \qquad (20)$$

where $\beta(n)$ has the continued fraction expansion

$$\beta(n) = [\overline{a_1(n), \ldots, a_R(n)}].$$

This means that $\beta(n)$ satisfies the quadratic equation

$$\beta(n) = [a_1(n), \ldots, a_R(n), \beta(n)],$$

that can be rewritten as

$$q'_R(n)\beta(n)^2 + (q'_{R-1}(n) - p'_R(n))\beta(n) - p'_{R-1}(n) = 0, \qquad (21)$$

where $p'_i(n)/q'_i(n) = [a_1(n), \ldots, a_i(n)]$.

This means that the integers $p'_{R-1}(n)$, $p'_R(n)$, $q'_{R-1}(n)$ and $q'_R(n)$ appearing in (21) are all $\ll (\max_{1 \leq i \leq R} a_i(n))^R$.

From (19) it follows that $\max_{1\le i\le R} a_i(n) < e^{n\delta}$, which implies that $p'_{R-1}(n),\ p'_R(n),\ q'_{R-1}(n)$ and $q'_R(n)$ are all $\ll e^{Rn\delta}$.

Taking the trace of both terms of (20) we get that for infinitely many n

$$2a_0(n) = \frac{q'_{R-1}(n) - p'_R(n)}{p'_{R-1}(n)}. \tag{22}$$

Estimating the height on both sides of (22), on the left side we get

$$H(2a_0(n)) = 2a_0(n) = 2\lfloor\sqrt{\alpha(n)}\rfloor \gg 2^{n/2}$$

(since α can be supposed a non-constant power sum), while on the right side we have

$$H\Big(\frac{q'_{R-1}(n) - p'_R(n)}{p'_{R-1}(n)}\Big) \ll \max\{q'_{R-1}(n), p'_R(n), p'_{R-1}(n)\} \ll\ e^{Rn\delta}$$

(since $q'_{R-1}(n),\ p'_R(n),$ and $p'_{R-1}(n)$ are integers), getting a contradiction choosing $\delta < \dfrac{\ln 2}{2R}$, i.e. $\epsilon < \dfrac{\ln 2}{2(k+1)R}$.

□

Proof of the Main Theorem 3.4. The case of α constant is trivial; thus we can suppose α to be non constant for the rest of the proof.

For $r \in \{0,1\}$ fixed, let

$$\sqrt{\alpha(2m+r)} = [a_0(m); a_1(m), a_2(m), \ldots] = [a_0(m); \overline{a_1(m), \ldots, a_{R(m)}(m)}]$$

be the continued fraction expansion for $\sqrt{\alpha(2m+r)}$, and let $p_i(m)/q_i(m)$, $i = 0,1,\ldots,$ with $q_0(m) = 1$, be the (infinite) sequence of its convergents. If $m \in A$, we have $R(m) = R$.

We recall that the relations $a_R(m) = 2a_0(m)$, for every $m \in A$ (if $R > 0$), and

$$a_{i+1}(m) < \Big|\sqrt{\alpha(2m+r)} - \frac{p_i(m)}{q_i(m)}\Big|^{-1} q_i(m)^{-2}, \tag{23}$$

for every $i \ge 0$ and $m \in \mathbb{N}$, hold.

By our present assumption, the hypothesis of Corollary 3.3 cannot hold for α and for the fixed r, since the period of the continued fraction for $\sqrt{\alpha(n)}$ cannot

tend to infinity for $n \longrightarrow +\infty$. This means that for a certain $\rho > 0$, there exists a power sum $\eta \in \Sigma$ such that

$$|\alpha(2m+r) - \eta(m)^2| \ll \alpha(2m+r)^{1/2-\rho}. \tag{24}$$

From (24) it follows

$$|\sqrt{\alpha(2m+r)} - \eta(m)| \ll \alpha(2m+r)^{-\rho} < 1, \tag{25}$$

the last inequality holding for $m \in \mathbb{N}$ large. Since α has integral coefficients, there exists η satisfying (25) having the same property; this means that $\eta(m)$ is an integer for every m. Since $\eta(m)$ is an integer and since (25) holds, it follows that

$$a_0(m) = \lfloor \sqrt{\alpha(2m+r)} \rfloor \in \{\eta(m),\ \eta(m)-1\} \tag{26}$$

for every $m \in \mathbb{N}$ large enough.

We claim that either $a_0(m) = \eta(m)$ or $a_0(m) = \eta(m) - 1$ for all $m \in \mathbb{N}$ large enough. In fact, $a_0(m) = \eta(m)$ when $\alpha(2m+r) - \eta(m)^2 \geq 0$, while $a_0(m) = \eta(m) - 1$ when $\alpha(2m+r) - \eta(m)^2 < 0$, and just one of the above inequalities can hold for all m large, since α and η are power sums. This proves that for $m \in \mathbb{N}$ large enough $a_0(m)$ is a power sum in $\mathbb{Z}\Sigma_{\mathbb{Z}}$.

If $R = 0$, the proof is complete.

Note that since α was supposed to be non constant, also $a_o(m)$ is non constant.

Consider from now on $R > 0$, and suppose by contradiction that there exists $h \in \mathbb{N}$, $1 \leq h \leq R$, such that for $m \in A$ large enough, $a_i(m)$ can be parameterized by a power sum in $\mathbb{Z}\Sigma_{\mathbb{Z}}$ for $i = 0, \ldots, h-1$, but not for $i = h$. The case $h = R$ can be excluded, since for $m \in A$ we have $a_R(m) = 2a_0(m) \in \mathbb{Z}\Sigma_{\mathbb{Z}}$.

Put $a(m) := [a_0(m); a_1(m), \ldots, a_{h-1}(m)] = \dfrac{p_{h-1}(m)}{q_{h-1}(m)} \in \mathbb{Q}$.

Since $a_i(m) \in \mathbb{Z}\Sigma_{\mathbb{Z}}$ for every $i = 0, \ldots, h-1$, the relation

$$|\sqrt{\alpha(2m+r)} - a(m)|^{-1} = \frac{\sqrt{\gamma(m)} + \tau(m)}{\xi(m)} =: \alpha_h(m) \tag{27}$$

holds for every $m \in A$ large enough, and for certain power sums γ, τ and $\xi \in \mathbb{Z}\Sigma_{\mathbb{Z}}$, ξ not identically zero.

We claim that for every $\epsilon > 0$ there does not exist a power sum $\zeta \in \Sigma$ such that

$$\Big|\alpha_h(m) - \zeta(m)\Big| \ll e^{-(2m+r)\epsilon}. \tag{28}$$

In fact, if such a power sum would exist, in view of (28), we would have

$$\Big|\alpha_h(m) - \zeta(m)\Big| < 1$$

for $m \in A$ large enough, which implies that

$$a_h(m) = \lfloor \alpha_h(m) \rfloor \in \{\lfloor \zeta(m) \rfloor - 2,\ \lfloor \zeta(m) \rfloor - 1,\ \lfloor \zeta(m) \rfloor\},$$

for $m \in A$ large enough. But since ζ has integral roots and rational coefficients, there exist arithmetic progressions $A_s = \{m = tm' + s,\ m' \in \mathbb{N}\}$, for $s = 0, \ldots, t-1$ and some $t \in \mathbb{N}$, such that $\lfloor \zeta(m) \rfloor$ can be parameterized by a power sum in $\mathbb{Z}\Sigma_{\mathbb{Z}}$ for all $m \in A$ in any of such progressions. Choose a progression, say A_1, that contains infinitely many elements $m \in A$. Let us notice that the set A in the statement of the present Theorem can be substituted without losing generality by any of its infinite subsets (A is just an infinite set for which $R(m) = R$, and not the set of all m for which $R(m) = R$). Substituting the set A in the statement of the present Theorem by the (still infinite) set $A \cap A_1$, which for simplicity of notation we will call A again, we would get that for all $m \in A$ large enough, $a_h(m)$ can be parameterized by a power sum in $\mathbb{Z}\Sigma_{\mathbb{Z}}$, a contradiction proving that α_h satisfies the assumption of Theorem 3.1.

By the definition of $\alpha_h(m)$, the length of the period of its continued fraction is R again. Let

$$\alpha_h(m) = [a'_0(m); \overline{a'_1(m), \ldots, a'_R(m)}],$$

and let $p'_i(m)/q'_i(m)$, $i = 0, 1, \ldots,$ with $q'_0(m) = 1$, be the (infinite) sequence of its convergents.

We have the relations $a'_i(m) = a_{i+h}(m)$ for $i + h \leq R$, $a'_i(m) = a_{i+h-R}(m)$ for $i + h > R$, and

$$a'_{i+1}(m) < \left|\alpha_h(m) - \frac{p'_i(m)}{q'_i(m)}\right|^{-1} \tag{29}$$

for every $i \geq 0$.

Since α_h satisfies the assumption for Theorem 3.1, for some $\epsilon > 0$ to be fixed later there exist $k \geq 3$ and $Q = e^{\epsilon} > 1$ as in that statement.

As in the proof of Corollary 3.3, we have again the inequality $q'_i(m) < Q^{2m+r}$, which holds for every $i = 0, \dots, R$ and m large, i.e. we can apply Theorem 3.1 to $\alpha_h(m)$ with $p = p'_i(m)$, $q = q'_i(m)$ and some $\epsilon > 0$ to be fixed later. We get that for every $i \geq 0$ and for $m \in A$ large enough,

$$\left|\alpha_h(m) - \frac{p'_i(m)}{q'_i(m)}\right| \geq q'_i(m)^{-k} e^{-(2m+r)\epsilon}. \tag{30}$$

Recalling that $0 < q'_i(m) < Q^{2m+r} = e^{(2m+r)\epsilon}$, for every $i = 0, \dots, R$, and considering the inequality (30) for $i = R - h - 1$, together with (29), we have

$$a_R(m) = a'_{R-h}(m) \leq \left|\alpha_h(m) - \frac{p'_{R-h-1}(m)}{q'_{R-h-1}(m)}\right|^{-1} \leq q'_{R-h-1}(m)^k \ e^{(2m+r)\epsilon} <$$

$$< Q^{(2m+r)k} e^{(2m+r)\epsilon} = e^{(2m+r)(k+1)\epsilon} = e^{(2m+r)\epsilon'}, \tag{31}$$

for $\epsilon' = (k+1)\epsilon$.

Choosing $\epsilon < \frac{\ln 2}{2(k+1)}$ (i.e. $\epsilon' < \frac{\ln 2}{2}$), we get that

$$a_R(m) \ll 2^{m(1-\delta)},$$

for some $\delta > 0$.

Recalling that $a_0(m) \in \mathbb{Z}\Sigma_{\mathbb{Z}}$ is non constant, from the relation

$$a_R(m) = 2a_0(m) \gg 2^m$$

we get a contradiction, proving that the relation (5) holds for every $m \in A$, except finitely many.

It remains to show that (5) holds for every $m \in \mathbb{N}$, except finitely many. We will proceed by contradiction.

We have already proved that $a_0(m) = \beta_0(m)$ for every $m \in \mathbb{N}$ large enough.

Suppose that for some $u > 0$, $a_i(m) = \beta_i(m)$ for every $i = 0, \ldots, u-1$ and for every $m \in \mathbb{N}$ except finitely many, but $a_u(m) \neq \beta_u(m)$ for infinitely many $m \in \mathbb{N}$ (we define $\beta_{aR+b}(m) := \beta_b(m)$, for $a \in \mathbb{N}$ and $0 \leq b < R$).

Let $a'(m) := [\beta_0(m), \ldots, \beta_{u-1}(m)]$.

We know that for $m \in \mathbb{N}$ large enough,

$$|\sqrt{\alpha(2m+r)} - a'(m)|^{-1} = \frac{\sqrt{\gamma'(m)} + \tau'(m)}{\xi'(m)},$$

for certain $\gamma', \eta', \xi' \in \mathbb{Z}\Sigma_{\mathbb{Z}}$, ξ' not identically zero.

For $m \in A$ large enough we have

$$\beta_u(m) = a_u(m) = \lfloor |\sqrt{\alpha(2m+r)} - a'(m)|^{-1} \rfloor = \left\lfloor \frac{\sqrt{\gamma'(m)} + \tau'(m)}{\xi'(m)} \right\rfloor, \quad (32)$$

which means that both the inequalities

$$\frac{\sqrt{\gamma'(m)} + \tau'(m)}{\xi'(m)} - \beta_u(m) \geq 0 \quad (33)$$

and

$$\frac{\sqrt{\gamma'(m)} + \tau'(m)}{\xi'(m)} - \beta_u(m) < 1 \quad (34)$$

hold for $m \in A$ large enough.

The inequalities (33) and (34) can be rewritten as

$$\gamma'(m) - (\beta_u(m)\xi'(m) - \tau'(m))^2 \geq 0 \quad (35)$$

and

$$\gamma'(m) - (\xi'(m) + \xi'(m)\beta_u(m) - \tau'(m))^2 < 0 \quad (36)$$

respectively.

Since $\beta_u, \gamma', \tau', \xi' \in \mathbb{Z}\Sigma_{\mathbb{Z}}$ are power sums, both the inequalities (35) and (36) can hold either for every $m \in \mathbb{N}$ except finitely many, or just for a finite set of m. Since we know that they hold for an infinite subset of A, they must hold

for every $m \in \mathbb{N}$, except at most finitely many, i.e. $\beta_u(m) = a_u(m)$ for every $m \in \mathbb{N}$ except finitely many, a contradiction proving that

$$\sqrt{\alpha(2m+r)} = [\beta_0(m); \overline{\beta_1(m), \ldots, \beta_R(m)}]$$

for every $m \in \mathbb{N}$, apart from finitely many exceptions.

$\square$

Acknowledgement: The author is grateful to Prof. U. Zannier for bringing this problem to his attention and to Prof. P. Corvaja for valuable comments and remarks.

References

[1] Y. Bugeaud and F. Luca, *On the period of the continued fraction expansion of* $\sqrt{2^{2n+1}+1}$. Preprint (2004), available at http://arxiv.org/PS_cache/math/pdf/0401/0401341.pdf.

[2] P. Corvaja and U. Zannier, *Diophantine Equations with Power Sums and Universal Hilbert Sets*. Indag. Math., N.S. (3) **9** (1998), 317–332.

[3] P. Corvaja and U. Zannier, *On the length of the continued fraction for the values of quotients of power sums*. J. Théor. Nombres Bordeaux, to appear (2004).

[4] L. K. Hua, *On the least solution to Pell's equation*. Bull. Amer. Math. Soc. **48** (1942), 731–735.

[5] A. Ya. Khinchin, *Continued fractions*. Dover Publ., 1997.

[6] H. W. Lenstra, Jr., *Solving the Pell equation*. Notices Amer. Math. Soc. (2) **49** (2002), 182–192.

[7] A. Schinzel, *On some problems of the arithmetical theory of continued fractions*. Acta Arith. **6** (1961), 393–413.

[8] A. Schinzel, *On some problems of the arithmetical theory of continued fractions II.* Acta Arith. **7** (1962), 287–298.

[9] W. M. Schmidt, *Diophantine Approximation.* Lecture Notes in Math. vol. 785. Springer-Verlag (1980).

[10] W. M. Schmidt, *Diophantine Approximations and Diophantine Equations.* Lecture Notes in Math. vol. 1467. Springer-Verlag (1991).

Amedeo Scremin
Institut für Mathematik
TU Graz
Steyrergasse 30
A-8010 Graz, Austria
e-mail: scremin@finanz.math.tu-graz.ac.at